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Abelian ideals of a Borel subalgebra and subsets of the Dynkin diagram

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Let $\mathfrak{g}$ be a simple Lie algebra and $\mathfrak{Ab}(\mathfrak{g})$ the set of abelian ideals of a Borel subalgebra of $\mathfrak{g}$. In this note, an interesting connection between $\mathfrak{Ab}(\mathfrak{g})$ and the subsets of the Dynkin diagram of $\mathfrak{g}$ is discussed. We notice that the number of abelian ideals with k generators equals the number of subsets of the Dynkin diagram with k connected components. For $\mathfrak{g}$ of type $\mathbf{A}_n$ or $\mathbf{C}_n$, we provide a combinatorial explanation of this coincidence by constructing a suitable bijection. We also construct a general bijection between $\mathfrak{Ab}(\mathfrak{g})$ and the subsets of the Dynkin diagram, which is based on the theory developed by Peterson and Kostant.

Introduction

Let $\mathfrak{g}$ be a complex simple Lie algebra with a Borel subalgebra $\mathfrak{b}$. The set of abelian ideals of $\mathfrak{b}$, denoted $\mathfrak{Ab}(\mathfrak{g})$, attracted much attention after appearance of [6], where Kostant popularised (and elaborated on) a remarkable result of D. Peterson to the effect that $\#\mathfrak{Ab}(\mathfrak{g}) = 2^{\operatorname{rk}\mathfrak{g}}$. The aim of this note is to report on a surprising connection between $\mathfrak{Ab}(\mathfrak{g})$ and the subsets of the Dynkin diagram of $\mathfrak{g}$. Namely, comparing independently performed computations [4,9], we notice that the number of abelian ideals with k generators equals the number of subsets of the Dynkin diagram with k connected components (see details in Section 1). For $\mathfrak{g}$ of type $\mathbf{A}_n$ or $\mathbf{C}_n$, we provide a combinatorial explanation of this coincidence by constructing a suitable bijection between $\mathfrak{Ab}(\mathfrak{g})$ and the subsets of the Dynkin diagram (see Section 2). In Section 3, we construct a general bijection between $\mathfrak{Ab}(\mathfrak{g})$

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and the subsets of the Dynkin diagram. Although this last bijection does not respect the number of generators and connected components, we believe it is interesting in its own right. This exploits a relationship between the abelian ideals and certain elements of the affine Weyl group of $\mathfrak{g}$ [6].

We refer to [5] for standard results on root systems and affine Weyl groups.

1. An empirical observation

Let Δ be the root system of $(\mathfrak{g}, \mathfrak{t})$, where $\mathfrak{t}$ is any Cartan subalgebra contained in $\mathfrak{b}$, and Δ^+ the subset of positive roots corresponding to $\mathfrak{b}$. Then $\Pi = \{\alpha_1, \dots, \alpha_n\}$ is the set of simple roots in Δ^+. We regard Δ^+ as poset with respect to the *root order*. This means that $\nu \preccurlyeq \mu$ if $\mu - \nu$ is a non-negative integral linear combination of simple roots.

An ideal $\mathfrak{a}$ of $\mathfrak{b}$ is said to be *abelian*, if $[\mathfrak{a}, \mathfrak{a}] = 0$. Then $\mathfrak{a}$ is a sum of certain root spaces in $\mathfrak{u} = [\mathfrak{b}, \mathfrak{b}]$, i.e., $\mathfrak{a} = \bigoplus_{\gamma \in I} \mathfrak{g}_\gamma$. Here I is necessarily an *upper ideal* of Δ^+, i.e., if $\nu \in I$, $\mu \in \Delta^+$, and $\nu + \mu \in \Delta^+$, then $\nu + \mu \in I$. In other words, if $\nu \in I$, $\gamma \in \Delta^+$, and $\nu \preccurlyeq \gamma$, then $\gamma \in I$. The property of being abelian means that $\gamma' + \gamma'' \notin \Delta^+$ for all $\gamma', \gamma'' \in I$. Let $\mathfrak{Ab} = \mathfrak{Ab}(\mathfrak{g})$ be the poset, with respect to inclusion, of all abelian ideals. We will mostly work in the combinatorial setting, so that an abelian ideal $\mathfrak{a}$ is identified with the corresponding set I of positive roots. The minimal elements (roots) of I are also called the *generators* of I.

Let $\kappa(I)$ be the number of minimal elements of I. The generating function

$$\hat{\mathcal{K}}_{\mathfrak{Ab}}(q) := \sum_{I \in \mathfrak{Ab}} q^{\kappa(I)}$$

is called the *upper covering polynomial* (of the poset $\mathfrak{Ab}$). We refer to [9] for generalities on covering polynomials. In fact, there is also a *lower covering polynomial*, which is not considered here. The polynomials $\hat{\mathcal{K}}_{\mathfrak{Ab}(\mathfrak{g})}(q)$ are known for all simple Lie algebras $\mathfrak{g}$, see [8, Section 5] and [9, Section 5]. By the very definition, the coefficient of q^k is the number of abelian ideals with k generators.

Recently, we have discovered that the polynomials $\hat{\mathcal{K}}_{\mathfrak{Ab}(\mathfrak{g})}(q)$ had another interpretation in terms of the Dynkin diagram of Δ. Regarding Π as the set of nodes in the Dynkin diagram, we say that a subset of Π is *connected* if it is connected in the Dynkin diagram. Then, for any subset of Π, we can consider the number of its connected components. Let $N_k = N_k(\Delta)$ denote the number of subsets of Π with exactly k connected components. Then, of course, $\sum_{k \geqslant 0} N_k = 2^n$. For all Δ, the numbers N_k are found in [4, n° 2].[1] Comparing them with our upper covering polynomials, we get the striking assertion:

Theorem 1.1. *For any reduced irreducible root system Δ (i.e., for any simple Lie algebra $\mathfrak{g}$), we have*

$$\sum_{k \geqslant 0} N_k q^k = \hat{\mathcal{K}}_{\mathfrak{Ab}(\mathfrak{g})}(q).$$

In other words, the number of abelian ideals with k generators equals the number of subsets of Π with k connected components.

Actually, one of the goals of [4] is to classify the closed subsets P of Δ such that $\Delta \setminus P$ is also closed. Such a P is said to be *invertible*. If P is invertible, then so is $w(P)$ for any $w \in W$. Let $N(\Delta)$ be the number of W-orbits in the set of all invertible subsets of Δ. It is shown in [4, Eq. (2)] that

$$N(\Delta) = \sum_{k \geqslant 0} N_k 2^k.$$

[1] More precisely, there is only a recursive formula for $N_k(\mathbf{D}_n)$ in [4]. But it is equivalent to the recursive formula for polynomials $\hat{\mathcal{K}}_{\mathfrak{Ab}}(q)$, cf. [9, Eq. (5.1)] and (1.1) below.

Table 1
The upper covering polynomials for $\mathfrak{Ab}(\mathfrak{g})$.

$\mathfrak{g}$	$\hat{\mathcal{K}}_{\mathfrak{Ab}(\mathfrak{g})}(q)$
$\mathbf{A}_n$, $\mathbf{B}_n$, $\mathbf{C}_n$	$\sum_{k\geqslant 0}\binom{n+1}{2k}q^k$
$\mathbf{D}_n$	$\sum_{k\geqslant 0}(\binom{n+2}{2k}-4\binom{n-1}{2k-2})q^k$
$\mathbf{E}_6$	$1+25q+27q^2+11q^3$
$\mathbf{E}_7$	$1+34q+60q^2+30q^3+3q^4$
$\mathbf{E}_8$	$1+44q+118q^2+76q^3+17q^4$
$\mathbf{F}_4$	$1+10q+5q^2$
$\mathbf{G}_2$	$1+3q$

In this way, one obtains a surprising interpretation of the value $\hat{\mathcal{K}}_{\mathfrak{Ab}(\mathfrak{g})}(2)$. For the reader's convenience, we reproduce a table with all these polynomials.

In [9, Section 5], we observed that if the Dynkin diagram has no branching nodes, then $\hat{\mathcal{K}}_{\mathfrak{Ab}(\mathfrak{g})}$ depends only on $\mathsf{rk}(\mathfrak{g})$, i.e., on the number of nodes. For instance, the upper covering polynomial for $\mathbf{F}_4$ (resp. $\mathbf{G}_2$) is equal to that for $\mathbf{A}_4$ (resp. $\mathbf{A}_2$). Having at hand Theorem 1.1, we now realise that the reason is that the connected components of a subset of Π does not depend on the length of simple roots.

There are some regularities in Table 1. For all classical series, these polynomials satisfy the recurrence relation

$$\hat{\mathcal{K}}_{\mathfrak{Ab}(\mathbf{X}_n)}(q)=2\hat{\mathcal{K}}_{\mathfrak{Ab}(\mathbf{X}_{n-1})}(q)+(q-1)\hat{\mathcal{K}}_{\mathfrak{Ab}(\mathbf{X}_{n-2})}(q), \tag{1.1}$$

where $\mathbf{X}\in\{\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}\}$. Furthermore, the sequence $\mathbf{E}_3=\mathbf{A}_2\times\mathbf{A}_1$, $\mathbf{E}_4=\mathbf{A}_4$, $\mathbf{E}_5=\mathbf{D}_5$, $\mathbf{E}_6$, $\mathbf{E}_7$, $\mathbf{E}_8$ can be regarded as the 'exceptional' series, and for this series the same recurrence relation holds. Comparing the coefficients of q^k in (1.1), one obtains the relation

$$N_k(\mathbf{X}_n)=2N_k(\mathbf{X}_{n-1})+N_{k-1}(\mathbf{X}_{n-2})-N_k(\mathbf{X}_{n-2}). \tag{1.2}$$

That is, it is true not only for $\mathbf{D}_n$, as pointed out in [4, p. 341], but for all our series, including the exceptional one. Actually, relation (1.2) for the number of subsets with prescribed number of connected components remains true if we extend any finite graph $\mathcal{G}_{n-2}$ with a chain of length 2, see the pattern below:

$$\mathcal{G}_n:\quad \circ\!-\!\underbrace{\circ - \boxed{\mathcal{G}_{n-2}}}_{\mathcal{G}_{n-1}}$$

We leave it to the reader to prove (1.2) for $\mathbf{X}_n=\mathcal{G}_n$.

Remark 1.2. For a sequence of polynomials $\mathcal{K}_n(q)$ satisfying relation (1.1), we have $\mathcal{K}_n(-1)=2\mathcal{K}_{n-1}(-1)-2\mathcal{K}_{n-2}(-1)$. This yields a kind of 4-periodicity for the values at $q=-1$: $\mathcal{K}_{n+2}(-1)=-4\mathcal{K}_{n-2}(-1)$.

2. A good bijection for $\mathbf{A}_n$ and $\mathbf{C}_n$

In what follows, we write 2^Π for the set of all subsets of Π. Theorem 1.1 suggests that there could be a natural one-to-one correspondence between $\mathfrak{Ab}(\mathfrak{g})$ and 2^Π, under which the ideals with k generators correspond to the subsets with k connected components. We call it a *good bijection*. So far, we did not succeed in finding such a good bijection in general. In fact, we are able to construct a general bijection $\mathfrak{Ab}(\mathfrak{g})\xrightarrow{1:1}2^\Pi$ (see Section 3), but that bijection is not good.

In this section, a good bijection is constructed for $\mathfrak{g} = \mathfrak{sl}_{n+1}$ or $\mathfrak{sp}_{2n}$.

Let $\Pi = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid i = 1, 2, \dots, n\}$ be the standard set of simple roots for $\mathbf{A}_n$. We regard Π as the n-element interval: $[n] := \{1, 2, \dots, n\}$. Every positive root γ of $\mathbf{A}_n$ is of the form $\gamma = \alpha_i + \alpha_{i+1} + \dots + \alpha_j$ with $i \leqslant j$, and therefore we identify it with the subset (*interval*) $[i, j] := \{i, i+1, \dots, j\}$ of $[n]$. In the usual terminology on root systems, $[i, j]$ is the *support* of γ, also denoted $\mathrm{supp}(\gamma)$.

Now, let I be an abelian ideal in $\Delta^+(\mathbf{A}_n)$ and $\gamma_1, \dots, \gamma_k$ the set of generators of I. Then such an ideal is also denoted by $I(\gamma_1, \dots, \gamma_k)$. (Of course, this imposes certain restrictions on γ_i's, which we describe below.)

Let $\Phi : \mathfrak{Ab}(\mathfrak{sl}_{n+1}) \to \{\text{subsets of } [n]\} =: 2^{[n]}$ be defined by the formula:

$$I = I(\gamma_1, \dots, \gamma_k) \mapsto \mathrm{supp}(\gamma_1) \oplus \dots \oplus \mathrm{supp}(\gamma_k),$$

where '$\oplus$' stands for the "exclusive disjunction" (or "addition mod 2") in the Boolean algebra of subsets of $[n]$.

Theorem 2.1. *The map Φ sets up a one-to-one correspondence between $\mathfrak{Ab}(\mathfrak{sl}_{n+1})$ and $2^{[n]}$. Moreover, if I has k generators, then $\Phi(I)$ has k connected components.*

Proof. Suppose we are given k positive roots $\gamma_s = [i_s, j_s]$, $s = 1, \dots, k$. Without loss of generality, we can assume that $i_1 \leqslant i_2 \leqslant \dots \leqslant i_k$. It is then easily seen that $\{\gamma_1, \dots, \gamma_k\}$ is the set of generators of an abelian ideal if and only if

$$1 \leqslant i_1 < i_2 < \dots < i_k \leqslant j_1 < j_2 < \dots < j_k \leqslant n. \tag{2.1}$$

(This presentation also shows that $\#\mathfrak{Ab}(\mathfrak{sl}_{n+1}) = 2^n$.) Thus, we have $2k$ points (or $2k - 1$ if $i_k = j_1$) in the whole interval $[n]$. Now, a straightforward verification shows that $[i_1, j_1] \oplus \dots \oplus [i_k, j_k]$ is the union of the following intervals:

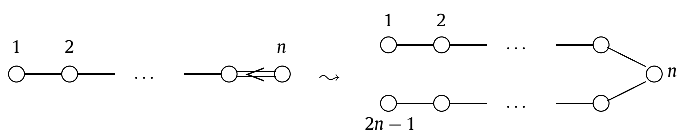
- we begin with the interlacing i-intervals: $[i_1, i_2 - 1], [i_3, i_4 - 1], \dots$;
- we end up with the interlacing j-intervals: $\dots, [j_{k-3} + 1, j_{k-2}], [j_{k-1} + 1, j_k]$;
- if k is odd, then we also take the middle interval $[i_k, j_1]$.

The total number of such intervals equals k, as required. Note that these intervals are disjoint and, moreover, each interval is a connected component of their union. Conversely, any collection of k such intervals allows us to write up a sequence of the form (2.1) and obtain an abelian ideal. $\square$

Example. For $k = 3$, we obtain the intervals $[i_1, i_2 - 1]$, $[i_3, j_1]$, $[j_2 + 1, j_3]$.

For $k = 4$, we obtain the intervals $[i_1, i_2 - 1]$, $[i_3, i_4 - 1]$, $[j_1 + 1, j_2]$, $[j_3 + 1, j_4]$.

To construct a good bijection for $\mathfrak{g} = \mathfrak{sp}_{2n}$, we use the usual unfolding $\mathbf{C}_n \rightsquigarrow \mathbf{A}_{2n-1}$ (see figure below) and combine it with the above $\mathfrak{sl}$-algorithm.



If Δ is of type $\mathbf{C}_n$, then $\Pi = \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n, 2\varepsilon_n\}$ and the unique maximal abelian ideal consists of the roots $\{\varepsilon_i + \varepsilon_j \mid 1 \leqslant i \leqslant j \leqslant n\}$. The positive roots of $\mathbf{A}_{2n-1}$ are identified with the intervals of $[2n - 1]$, as above. Under the above unfolding, a short root $\varepsilon_i + \varepsilon_j$ $(i \neq j)$ is replaced with two roots $[i, 2n - j]$ and $[j, 2n - i]$ of $\mathbf{A}_{2n-1}$; and a long root $2\varepsilon_i$ is replaced with one root $[i, 2n - i]$.

If $I = I(\gamma_1, \dots, \gamma_k)$ is an abelian ideal of $\Delta^+(\mathbf{C}_n)$, then we do the following:

(i) Replace each γ_i with one or two roots (intervals) for $\mathbf{A}_{2n-1}$, as explained.
(ii) Take the sum modulo 2 of all these intervals. Obviously, the resulting subset of $[2n-1]$, $\tilde{\Phi}(I)$, is symmetric with respect to the middle point $\{n\}$.
(iii) Take the quotient of $\tilde{\Phi}(I)$ by this symmetry, i.e., consider $\Phi(I) := \tilde{\Phi}(I) \cap [n]$.

In this way, we obtain a mapping $\Phi : \mathfrak{Ab}(\mathfrak{sp}_{2n}) \to 2^{[n]}$, and it is not hard to verify that it is a good bijection.

3. A general bijection

In this section, a general bijection between $\mathfrak{Ab}(\mathfrak{g})$ and 2^{Π} is constructed. To this end, we need a parametrisation of the abelian ideals described by Kostant [6], which relies on the relationship, due to D. Peterson, between the abelian ideals and the so-called *minuscule elements* of the affine Weyl group of Δ. Recall the necessary setup.

We have the real vector space $V = \bigoplus_{i=1}^n \mathbb{R}\alpha_i$, the usual Weyl group generated by the reflections $s_1, \dots, s_n$, and a W-invariant inner product $(\ ,\ )$ on V. Then

$Q = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i \subset V$ is the root lattice;
$Q^+ = \{\sum_{i=1}^n m_i\alpha_i \mid m_i = 0, 1, 2, \dots\}$ is the monoid generated by the positive roots.

Letting $\widehat{V} = V \oplus \mathbb{R}\delta \oplus \mathbb{R}\lambda$, we extend the inner product $(\ ,\ )$ on $\widehat{V}$ so that $(\delta, V) = (\lambda, V) = (\delta, \delta) = (\lambda, \lambda) = 0$ and $(\delta, \lambda) = 1$. Set $\alpha_0 = \delta - \theta$, where θ is the highest root in Δ^+. Then

$\widehat{\Delta} = \{\Delta + k\delta \mid k \in \mathbb{Z}\}$ is the set of affine (real) roots;
$\widehat{\Delta}^+ = \Delta^+ \cup \{\Delta + k\delta \mid k \geqslant 1\}$ is the set of positive affine roots;
$\widehat{\Pi} = \Pi \cup \{\alpha_0\}$ is the corresponding set of affine simple roots.

As usual, set $\mu^\vee = 2\mu/(\mu, \mu)$ for $\mu \in \widehat{\Delta}$; $Q^\vee = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i^\vee$ is the coroot lattice in V.

For each $\alpha_i \in \widehat{\Pi}$, let s_i denote the corresponding reflection in $GL(\widehat{V})$. That is, $s_i(x) = x - (x, \alpha_i)\alpha_i^\vee$ for any $x \in \widehat{V}$. The affine Weyl group, $\widehat{W}$, is the subgroup of $GL(\widehat{V})$ generated by the reflections $s_0, s_1, \dots, s_n$. The inner product $(\ ,\ )$ on $\widehat{V}$ is $\widehat{W}$-invariant. For $w \in \widehat{W}$, we set $\mathcal{N}(w) = \{\nu \in \widehat{\Delta}^+ \mid w(\nu) \in -\widehat{\Delta}^+\}$.

Following D. Peterson, we say that $w \in \widehat{W}$ is *minuscule*, if $\mathcal{N}(w)$ is of the form $\{\delta - \gamma \mid \gamma \in I_w\}$ for some subset $I_w \subset \Delta$. It is not hard to prove that (i) $I_w \subset \Delta^+$, (ii) I_w is an abelian ideal, and (iii) the assignment $w \mapsto I_w$ yields a bijection between the minuscule elements of $\widehat{W}$ and the abelian ideals, see [6], [1, Proposition 2.8]. Conversely, if $I \in \mathfrak{Ab}$, then w_I stands for the corresponding minuscule element of $\widehat{W}$.

(I) The first step is to assign an element of $Q^\vee$ to an abelian ideal (i.e., to a minuscule element). This is known and, moreover, such an assignment can be performed for any ad-nilpotent ideal of $\mathfrak{b}$ [2]. In fact, one can associate an element of $Q^\vee$ to any $w \in \widehat{W}$. The following can be found in a more comprehensive form in [7, Section 2].

Recall that $\widehat{W}$ is a semi-direct product of W and $Q^\vee$, and it can be regarded as a group of affine-linear transformations of V [5]. For any $w \in \widehat{W}$, there is a unique decomposition

$$w = v \cdot t_r, \tag{3.1}$$

where $v \in W$ and t_r is the translation of V corresponding to $r \in Q^\vee$, i.e., $t_r * x = x + r$ for all $x \in V$. Then we assign the element $v(r) \in Q^\vee$ to $w \in \widehat{W}$. An alternative way for doing so, which does not explicitly use the semi-direct product structure, is based on the relation between the linear $\widehat{W}$-action on $\widehat{V}$ and decomposition (3.1). Define the integers k_i, $i = 1, \dots, n$, by the formula $w^{-1}(\alpha_i) = \mu_i + k_i\delta$

($\mu_i \in \Delta$). Then the element $v(r) \in Q^\vee$ is determined by the condition that $(v(r), \alpha_i) = k_i$. The reason is that $w^{-1} = v^{-1} \cdot t_{-v(r)}$ and the linear $\widehat{W}$-action satisfies the following relation

$$w^{-1}(x) = v^{-1}(x) + \big(x, v(r)\big)\delta \quad \forall x \in V \oplus \mathbb{R}\delta. \tag{3.2}$$

[It suffices to verify that $t_r(x) = x - (x, r)\delta$.]

(II) If $w = w_I$ is minuscule, then we also write z_I for the resulting element of $Q^\vee$. By [6, Theorem 2.5], the mapping $I \mapsto z_I \in V$ sets up a bijection between $\mathfrak{Ab}(\mathfrak{g})$ and $\mathcal{Z}_1 = \{z \in Q^\vee \mid (z, \gamma) \in \{-1, 0, 1, 2\}\ \forall \gamma \in \Delta^+\}$. This bijection is also mentioned in [3, Proposition 3.6]. Since neither [6] nor [3] contain a proof of this result, we provide a proof in Appendix A.

Having constructed $z_I \in Q^\vee$, we write

$$z_I = \sum_{i=1}^{n} m_i \alpha_i^\vee, \quad m_i \in \mathbb{Z}.$$

Finally, we define the subset S_I of Π as follows: $S_I = \{\alpha_i \in \Pi \mid m_i \text{ is odd}\}$. In other words, the Boolean vector $(m_1 \cdots m_n)$ (mod 2) is the characteristic vector of S_I.

Theorem 3.1. *The map $(I \in \mathfrak{Ab}(\mathfrak{g})) \mapsto (S_I \subset \Pi)$ sets up a one-to-one correspondence between $\mathfrak{Ab}(\mathfrak{g})$ and 2^Π.*

Proof. By a result of Peterson (see [6, Lemma 2.2]),

$$\mathcal{D} = \big\{x \in V \bigm| -1 < (x, \gamma) \leqslant 1\ \forall \gamma \in \Delta^+\big\}$$

is a fundamental domain for the $Q^\vee$-action on V by translations. Let $\zeta : V \to V/Q^\vee \simeq (\mathbb{S}^1)^n$ be the quotient map. Consider $\mathcal{Z}_1/2 \subset \mathcal{D}$. Clearly, $\mathcal{Z}_1/2 \to \zeta(\mathcal{Z}_1/2) \subset (\mathbb{S}^1)^n$ is bijective, and the image consists of all elements of order $\leqslant 2$. Equivalently, all the subsets S_I ($I \in \mathfrak{Ab}$) are different. $\square$

Unfortunately, this bijection does not behave well with respect to the number of generators and the number of connected components, see Example 3.3.

Remark 3.2. Let $\mathfrak{Par}(\mathfrak{g})$ be the set of all standard parabolic subalgebras of $\mathfrak{g}$. As is well known, there is a one-to-one correspondence

$$\mathfrak{Par}(\mathfrak{g}) \xleftrightarrow{1:1} 2^\Pi \tag{3.3}$$

that assigns to $\mathfrak{p} \in \mathfrak{Par}(\mathfrak{g})$ the set of simple roots of the standard Levi subalgebra of $\mathfrak{p}$.

On the other hand, there is a natural map $\Psi : \mathfrak{Ab}(\mathfrak{g}) \to \mathfrak{Par}(\mathfrak{g})$ that takes an abelian ideal $\mathfrak{a} \subset \mathfrak{b}$ to its normaliser in $\mathfrak{g}$, denoted $\mathfrak{n}_\mathfrak{g}(\mathfrak{a})$. This map was studied in [10], and it was proved there that Ψ is one-to-one *if and only if* $\mathfrak{g}$ is of type $\mathbf{A}_n$ or $\mathbf{C}_n$. In particular, combining Ψ with (3.3), we obtain the *third* natural bijection $\mathfrak{Ab}(\mathbf{A}_n) \to 2^\Pi$. It is remarkable that all three are different!

Example 3.3. For Δ of type $\mathbf{A}_3$, we compare three bijections given by Theorems 2.1, 3.1, and Remark 3.2. The first two columns of Table 2 contain the input: the vector $z_I \in Q^\vee$ and the set of generators of I. In the first (resp. second) column, a triple $m_1m_2m_3$ stands for $m_1\alpha_1^\vee + m_2\alpha_2^\vee + m_3\alpha_3^\vee$ (resp. $m_1\alpha_1 + m_2\alpha_2 + m_3\alpha_3$). The third column gives the characteristic vector of S_I. The last column shows the simple roots of the standard Levi subalgebra of $\mathfrak{n}_\mathfrak{g}(\mathfrak{a}_I)$, where $\mathfrak{a}_I$ is the ideal of $\mathfrak{b}$ corresponding to I. One sees that z_I (mod 2) and $\Phi(I)$ differ in the last two rows, and the last column is different from the previous two (even if we take the complement!).

Table 2
Three bijections for $\mathbf{A}_3$.

z_I	$\Gamma(I)$	z_I (mod 2)	$\Phi(I)$	Levi of $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{a}_I)$
000	$\varnothing$	000	$\varnothing$	{1,2,3}
111	{111}	111	{1,2,3}	{2}
110	{110}	110	{1,2}	{3}
011	{011}	011	{2,3}	{1}
100	{100}	100	{1}	{2,3}
001	{001}	001	{3}	{1,2}
010	{110, 011}	010	{1,3}	$\varnothing$
121	{010}	101	{2}	{1,3}

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Appendix A. A characterisation of minuscule elements

Here we prove the Kostant–Peterson characterisation of minuscule elements of $\widehat{W}$. Our proof exploits results of Cellini–Papi on arbitrary ad-nilpotent ideals. Let $\mathfrak{c}$ be an ad-nilpotent ideal of $\mathfrak{b}$, i.e., $\mathfrak{c} \subset \mathfrak{u}$ and $[\mathfrak{b}, \mathfrak{c}] \subset \mathfrak{c}$. By [1], there is a one-to-one correspondence between the ad-nilpotent ideals of $\mathfrak{b}$ and certain elements of $\widehat{W}$, which are said to be *minimal*. Write $\widehat{W}_{min}$ for the set of minimal elements. If $w \in \widehat{W}_{min}$ corresponds to $\mathfrak{c}$, then the set of roots of $\mathfrak{c}$ is $I_w = \{\gamma \in \Delta \mid w(\delta - \gamma) \in -\widehat{\Delta}^+\}$ and one can construct an element $v(r) \in Q^\vee$, as explained in the previous section.

Set $D_{min} = \{x \in V \mid (x, \alpha) \geqslant -1 \ \forall \alpha \in \Pi \ \& \ (x, \theta) \leqslant 2\}$. Using our notation, which differs from that of [1,2], results of Cellini–Papi can be stated as follows.

Theorem A.1. *(See [2, Propositions 2 and 3].)*

1. $w = v \cdot t_r \in \widehat{W}_{min} \iff \begin{cases} w(\alpha) \in \widehat{\Delta}^+ & \textit{for all } \alpha \in \Pi, \\ v(r) \in D_{min}. \end{cases}$
2. *The mapping* $\widehat{W}_{min} \to D_{min} \cap Q^\vee$, $w = v \cdot t_r \mapsto v(r)$, *is a bijection.*

The minuscule elements of $\widehat{W}$ are minimal, and the corresponding points $v(r)$ belong to a subset of D_{min}. This subset is described as follows.

Theorem A.2 *(Kostant–Peterson). Let* $w = v \cdot t_r \in \widehat{W}_{min}$. *Then*

$$w \textit{ is minuscule} \quad \textit{if and only if} \quad -1 \leqslant \big(\gamma, v(r)\big) \leqslant 2 \quad \textit{for all } \gamma \in \Delta^+.$$

Proof. 1. Suppose that w is minuscule. In particular,

$$\big(\alpha, v(r)\big) \geqslant -1 \quad \forall \alpha \in \Pi \quad \text{and} \quad \big(\theta, v(r)\big) \leqslant 2. \tag{A.1}$$

Assume that $(\tilde{\gamma}, v(r)) \leqslant -2$ for some $\tilde{\gamma} \in \Delta^+$. Then (A.1) implies that there is $\gamma \in \Delta^+$ such that $(\gamma, v(r)) = -2$. Arguing by induction on the height of γ, one readily proves that there are $\gamma_1, \gamma_2 \in \Delta^+$ such that $\gamma = \gamma_1 + \gamma_2$ and $(\gamma_i, v(r)) = -1$. By (3.2), we then have $w^{-1}(\gamma_i) = \mu_i - \delta$ for some $\mu_i \in \Delta$. Hence $w(\delta - \mu_i) = -\gamma_i \in -\Delta^+$ and $w(\delta - (\mu_1 + \mu_2)) = -\gamma - \delta \in -\widehat{\Delta}^+$. Therefore, $\mu_1, \mu_2, \mu_1 + \mu_2 \in I_w$, which contradicts the fact that I_w is abelian.

Assume that $(\tilde{\gamma}, v(r)) \geqslant 3$ for some $\tilde{\gamma} \in \Delta^+$. Then (A.1) implies that there is $\gamma \in \Delta^+$ such that $\tilde{\gamma} \preccurlyeq \gamma \preccurlyeq \theta$ and $(\gamma, v(r)) = 3$. Therefore $w^{-1}(\gamma) = -\mu + 3\delta$ for some $\mu \in \Delta$. Then $w(2\delta - \mu) = \gamma - \delta$ is a negative root and $2\delta - \mu \in \mathcal{N}(w)$, which contradicts the fact that w is minuscule.

2. Conversely, suppose that $-1 \leqslant (\gamma, v(r)) \leqslant 2$ for all $\gamma \in \Delta^+$, but w is not minuscule, i.e., I_w is not abelian. Then there exist $\mu_1, \mu_2 \in I_w$ such that $\mu_1 + \mu_2 = \mu \in \Delta^+$ and therefore

$$w(2\delta - \mu) = w\big((\delta - \mu_1) + (\delta - \mu_2)\big) \in -\widehat{\Delta}^+.$$

Write $w(2\delta - \mu) = -\nu - k\delta$ with $\nu \in \Delta$ and $k \geqslant 0$, and consider the possible values of k.

- If $k = 0$, then $\nu \in \Delta^+$. Then $w^{-1}(\nu) = \mu - 2\delta$, i.e., $(\nu, v(r)) = -2 < -1$. A contradiction!
- If $k \geqslant 1$, then $w^{-1}(-\nu) = -\mu + (k+2)\delta$. Hence $(-\nu, v(r)) = k + 2 \geqslant 3$, which contradicts the assumption whether ν is positive or negative. $\square$

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